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Technical Notes

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Differential View Factor for a Rectangle with Intervening Parallelepiped or Sphere

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Nomenclature

a, b	=	slope and z-intercept of straight-line segments
C	=	contour corresponding to the visible portion of rectangle on the projection plane
$\mathrm{d}A_1$	=	planar differential element
$F_{\mathrm{d}A1-2}$	=	view factor from differential element to finite rectangle
H	=	distance between differential element and center of the sphere
h	=	height from the base plane
l_1, m_1, n_1	=	direction cosines of normal to differential element
R	=	radius of the sphere
x, y, z	=	cartesian co-ordinates
θ	=	angular co-ordinate measured from x axis in anti-clockwise direction

Subscripts

i	=	corresponding to ith path
Pi	=	corresponding to ith point on the projection plane
1, 2, 3	=	corresponding to differential element, finite rectangle, and intervening object, respectively

Introduction

THERMAL radiation determines the temperatures attained by a surface in an enclosure having several objects, especially in space applications. Because of the geometrical complexities, the analytical determination of net radiation heat transfer from/to a surface is mathematically very difficult. To make the mathematical analysis feasible, the interior objects are often modeled into appropriate primitives such as parallelepipeds and spheres.[1] However, in spite of synthesizing the geometry and other simplifying assumptions, which are normally made in radiant-interchange calculations, it would be erroneous to assume that we could work with inaccurate values of view factors, as in enclosures such errors would be magnified.[2] Further, it has been shown that a minimum overall resolution of at least six significant digits is required in evaluating the view factors for several engineering applications.[3] Hence, the shadowing effect of intervening objects cannot be ignored while determining the view factors between two surfaces.

A review of literature shows that Katte and Venkateshan[4] presented analytical expressions for differential view factors in an axisymmetric enclosure with a coaxial cylinder inside by application of a contour integration method. Deiveegan et al.[5] presented analytical expressions for view factors between a differential element and a coaxial disk or rectangle when the receiving area was partially shadowed by an intervening disk or rectangle in a parallel plane. The comprehensive catalog of view factors[6] shows that analytical expressions for view factors are available only for a few configurations in the presence of intervening object or surface.

Hence, in the present work, closed form solutions are derived for the view factors from a planar differential area to a finite rectangle with an intervening parallelepiped or sphere when both the rectangle and intervening object are kept in a base plane arbitrarily. Three configurations are considered: 1) differential area is perpendicular to the base plane and the intervening object is a parallelepiped; 2) differential area is parallel to the base plane and the intervening object is a parallelepiped; and 3) differential area has an arbitrary orientation and the intervening object is a sphere.

Analysis

Parallelepiped as Intervening Object

Consider the view factor between differential element $\mathrm{d}A_1$ and a rectangle, with an intervening parallelepiped, as shown in Fig. 1, when both the rectangle and parallelepiped are placed arbitrarily on the base plane (parallel to x–y plane). The coordinate system is defined so that two direction cosines normal to $\mathrm{d}A_1$ are zeroes. Further, the corner with the least x coordinate is denoted as $(x_{2,1}, y_{2,1}, z_{2,1})$ on the rectangle and $(x_{3,1}, y_{3,1}, z_{3,1})$ on the parallelepiped, respectively. The other points are denoted following a counterclockwise direction if $h_1 < h_3$; otherwise a clockwise direction is followed.

As seen from $\mathrm{d}A_1$, the parallelepiped and rectangle are projected on a plane parallel to the x–z plane at unit distance, because it is easier to carry out the integration in the projected plane. Following Sparrow,[7] the view factor can be written as

$$F_{\mathrm{d}A1-2} = \frac{1}{2\pi}\left[m_1 \oint_C \frac{\bar{x}_2\,\mathrm{d}\bar{z}_2 - \bar{z}_2\,\mathrm{d}\bar{x}_2}{\bar{x}_2^2 + \bar{z}_2^2 + 1} + n_1 \oint_C \frac{\mathrm{d}\bar{x}_2}{\bar{x}_2^2 + \bar{z}_2^2 + 1} \right] \quad (1)$$

The intersection of various line segments of the projected rectangle with the projection of the intervening object constitutes the contour C. On the contour C, $\bar{y}_2$ remains constant; however, $\bar{x}_2$ and $\bar{z}_2$ vary, because the integration is carried out on the projected plane. Hence, a number of subcases are possible depending on the dimensions and positions of the rectangle and the intervening parallelepiped. For brevity, a general case is chosen, where the geometry of the shadow on the projection of the rectangle is hexagonal, as shown in Fig. 1, for which the contour C is given by P_1–P_5–P_{10}–P_9–P_8–P_7–P_6–P_2–P_3–P_4–P_1.

The co-ordinates of four corner points, P_1 to P_4, on the projection of the rectangle can be shown to be $x_{P1} = x_{P4} = x_{2,1}/y_{2,1}$; $x_{P2} = x_{P3} = x_{2,2}/y_{2,2}$; $z_{P1} = -h_1/y_{2,1}$; $z_{P2} = -h_1/y_{2,2}$; $z_{P3} = (h_2 - h_1)/y_{2,2}$; $z_{P4} = (h_2 - h_1)/y_{2,1}$. For the chosen case, the dimensions and position of parallelepiped have to satisfy all of the

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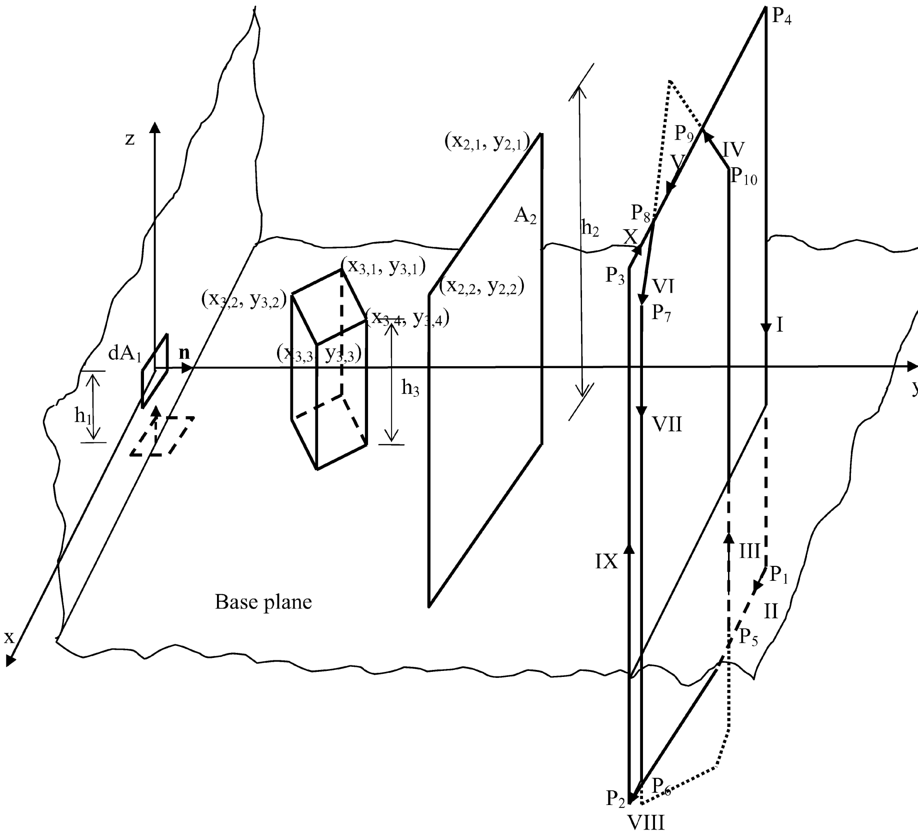

Fig. 1 Schematic diagram for the case of an intervening parallelepiped.

following constraints:

$$x_{2,1}/y_{2,1} < x_{3,k}/y_{3,k} < x_{2,2}/y_{2,2}, \qquad k = 1 \text{ and } 3$$

$$[(h_3 - h_1)/(h_1 - h_2)] < \{\lfloor x_{3,k}(y_{2,2} - y_{2,1}) - y_{3,k}(x_{2,2} - x_{2,1})\rfloor / (y_{2,1}x_{2,2} - y_{2,2}x_{2,1})\} < [(h_3 - h_1)/h_1], \qquad k = 1 \text{ and } 3$$

$$\{\lfloor x_{3,k}(y_{2,2} - y_{2,1}) - y_{3,k}(x_{2,2} - x_{2,1})\rfloor / (y_{2,1}x_{2,2} - y_{2,2}x_{2,1})\} < [(h_3 - h_1)/(h_1 - h_2)]$$
$$\text{if} \quad h_1 < h_3, \qquad k = 2 \text{ otherwise } k = 4 \tag{2}$$

In this case, a maximum of three corners of the top surface of the parallelepiped need to be projected to determine the geometry of the shadow. The coordinates of projections of these three points can be shown to be $x_{P8} = x_{3,k}/y_{3,k}$; $z_{P8} = (h_3 - h_1)/y_{3,k}$, if $h_1 < h_3, k = 2$, otherwise $k = 4$; $x_{P7} = x_{3,3}/y_{3,3}$; $z_{P7} = (h_3 - h_1)/y_{3,3}$; $x_{P10} = x_{3,1}/y_{3,1}$; $z_{P10} = (h_3 - h_1)/y_{3,1}$. The coordinates of the remaining points on the contour are determined by intersection of appropriate lines as

$$x_{P5} = x_{P10}; \qquad x_{P6} = x_{P7}$$

$$z_{Pi} = h_1 \lfloor x_{3,t}(y_{2,2} - y_{2,1}) - y_{3,t}(x_{2,2} - x_{2,1})\rfloor / \lfloor y_{3,t}(x_{2,2}y_{2,1} - y_{2,2}x_{2,1})\rfloor,$$
$$\text{for} \quad i = 5, t = 1 \quad \text{for} \quad i = 6, t = 3$$

For $i = 8$, $t = 1$, for $i = 9$, $t = 3$; $k = 2$ if $h_1 < h_3$, otherwise $k = 4$;

$$x_{Pi} = \frac{(h_2 - h_1)(x_{2,1} - x_{2,2})(x_{3,k}y_{3,t} - y_{3,k}x_{3,t}) - (h_3 - h_1)(x_{3,k} - x_{3,t})(x_{2,1}y_{2,2} - y_{2,1}x_{2,2})}{(h_3 - h_1)(y_{3,t} - y_{3,k})(x_{2,1}y_{2,2} - y_{2,1}x_{2,2}) - (h_2 - h_1)(y_{2,2} - y_{2,1})(x_{3,k}y_{3,t} - y_{3,k}x_{3,t})}$$

$$z_{pi} = \frac{h_2 - h_1}{x_{2,1}y_{2,2} - y_{2,1}x_{2,2}}[x_{2,1} - x_{2,2} + (y_{2,2} - y_{2,1})x_{PI}]$$

The direction of integration is shown in Fig. 1, and the visible contour is subdivided into 10 line segments. Thus Eq. (1) is represented as a sum of 10 line integrals along the contour. Because both $\bar{x}_2$ and $\bar{z}_2$ vary along the paths P_1–P_5, P_{10}–P_9, P_9–P_8, P_8–P_7, P_6–P_2, and P_3–P_4 (i.e., $i = 1$ to 6), on these segments $\bar{z}_2$ is represented in terms of $\bar{x}_2$ using equation of straight-line $\bar{z}_2 = a\bar{x}_2 + b$. Here slope a and $\bar{z}_2$-intercept b are determined by using the coordinates of respective end points of each segment. However, for the remaining segments (i.e., $i = 7$ to 10), $\bar{x}_2$ remains constant. Hence the solution of Eq. (1) on the path can be shown to be

$$F_{dA1-2} = \frac{1}{2\pi}\sum_{i=1}^{6}\frac{\delta_1 - (1-\delta_1)\,b_i}{\sqrt{a_i^2 + b_i^2 + 1}}\sum_{j=1}^{2}(-1)^j \times \tan^{-1}\left[\frac{\left(a_i^2+1\right)\bar{x}_2 + a_i b_i}{\sqrt{a_i^2 + b_i^2 + 1}}\right] + \frac{(1-\delta_1)}{2\pi} \times \sum_{i=7}^{10}\frac{x_{Pi-4}}{\sqrt{x_{Pi-4}^2 + 1}}\sum_{j=1}^{2}(-1)^j \tan^{-1}\left(\frac{\bar{z}_2}{\sqrt{x_{Pi-4}^2 + 1}}\right) \tag{3}$$

Here, the second summation in each term, represented by j, accounts for the limits of integration. The index $j = 1, 2$ denotes the lower and upper limits of integration, respectively, which are nothing but x or z coordinates of end points of the corresponding segment.

For case (i), because dA_1 is perpendicular to the base plane, $\delta_1 = 0$, and for case (ii), because dA_1 is parallel to the base plane, $\delta_1 = 1$.

If the parallelepiped does not intervene, the integrations along the paths P_1–P_5–P_{10}–P_9–P_8–P_7–P_6 in Eq. (3) vanish. Further, if the normal to dA_1 passes through a corner of rectangle, Eq. (3) for case 1 reduces to configuration B-3, available in the literature.[6] Similarly, the expression for case 2 reduces to configuration B-4.[6]

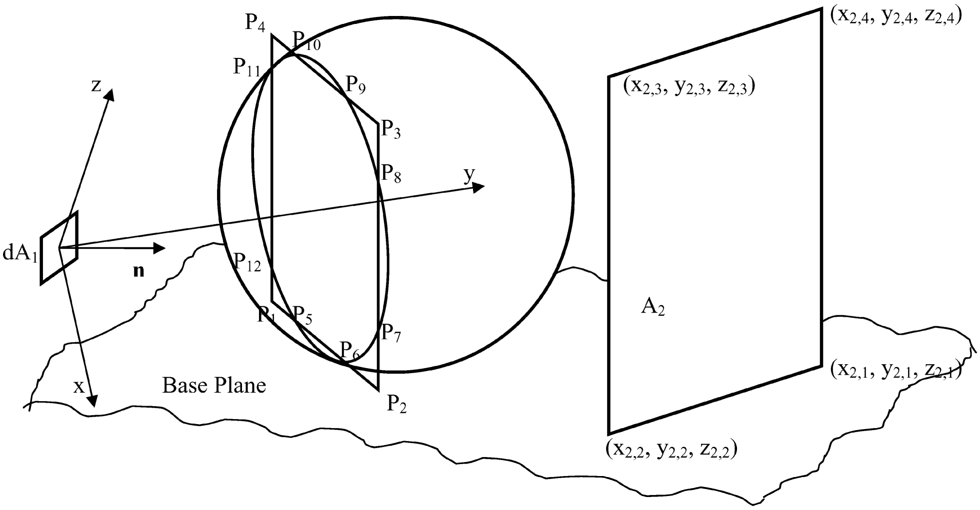

Fig. 2 Schematic diagram for the case of an intervening sphere.

Sphere as Intervening Object

Consider the view factor between dA_1 and a rectangle with an intervening sphere as shown in Fig. 2, when both the rectangle and sphere are placed arbitrarily on the base plane. The coordinate system, with its origin at dA_1, is chosen so that the line joining dA_1 and the center of the sphere coincides with the y-axis, if the plane containing dA_1 does not intersect the sphere. Otherwise, the projection of the line joining dA_1 and the center of the sphere onto the plane containing dA_1 coincides with the y-axis. As seen from dA_1, the sphere appears as a circle of radius $R\sqrt{(H^2-R^2)}/H$, with its center at $\lfloor 0, (H^2-R^2)/H, 0 \rfloor$. The rectangle is projected on the plane containing the circle, and the contour corresponding to the visible portion of rectangle on this plane is recognized. Following Sparrow,[7] the view factor for this case can be written as

$$F_{dA1-2} = \frac{1}{2\pi}\left[l_1 \oint_C \frac{\bar{z}_2\,d\bar{y}_2 - \bar{y}_2\,d\bar{z}_2}{\bar{x}_2^2+\bar{y}_2^2+\bar{z}_2^2} + m_1 \oint_C \frac{\bar{x}_2\,d\bar{z}_2 - \bar{z}_2\,d\bar{x}_2}{\bar{x}_2^2+\bar{y}_2^2+\bar{z}_2^2} + n_1 \oint_C \frac{\bar{y}_2\,d\bar{x}_2 - \bar{x}_2\,d\bar{y}_2}{\bar{x}_2^2+\bar{y}_2^2+\bar{z}_2^2} \right] \tag{4}$$

The intersection of various line segments of the projected rectangle with the circle constitutes the contour C. On C, $\bar{y}_2$ remains constant; however, $\bar{x}_2$ and $\bar{z}_2$ vary. For brevity, a general case is chosen, in which the circle intersects all the straight line segments of projection of the rectangle, as shown in Fig. 2, for which the contour C is given by P_4–P_1–P_5–P_{12}–P_{11}–P_{10}–P_9–P_8–P_7–P_6–P_2–P_3–P_4.

The coordinates of four corner points, P_1 to P_4, for the projection of the rectangle can be shown to be $x_{Pi} = x_{2,i}(H^2-R^2)/y_{2,i}H$; $z_{Pi} = z_{2,i}(H^2-R^2)/y_{2,i}H$, $i = 1$ to 4. For the chosen case, the dimensions and position of rectangle have to satisfy all of the following constraints:

$$n_1 \neq 1$$

$$(x_{2,4}z_{2,1} - z_{2,4}x_{2,1})/(y_{2,4}x_{2,1} - x_{2,4}y_{2,1}) > R\Big/\sqrt{H^2-R^2}$$

$$(x_{2,1}z_{2,2} - z_{2,1}x_{2,2})/(y_{2,1}x_{2,2} - x_{2,1}y_{2,2}) < R\Big/\sqrt{H^2-R^2}$$

$$(x_{2,3}z_{2,2} - z_{2,3}x_{2,2})/(y_{2,2}x_{2,3} - x_{2,2}y_{2,3}) > R\Big/\sqrt{H^2-R^2}$$

$$(x_{2,4}z_{2,3} - z_{2,4}x_{2,3})/(y_{2,3}x_{2,4} - x_{2,3}y_{2,4}) < R\Big/\sqrt{H^2-R^2} \tag{5}$$

The visible contour is subdivided into 12 segments; thus Eq. (4) is represented as a sum of 12 line integrals along the contour. Because both $\bar{x}_2$ and $\bar{z}_2$ vary along the paths P_4–P_1, P_1–P_5, P_{12}–P_{11}, P_{10}–P_9, P_8–P_7, P_6–P_2, P_2–P_3, and P_3–P_4, on these segments $\bar{z}_2$ is represented in terms of $\bar{x}_2$ using the equation of the straight line $\bar{z}_2 = a\bar{x}_2 + b$. However, for the remaining segments, polar coordinates are used to determine the angle of intersection θ, which is found by solving the equation of the circle and the corresponding line segment. Hence the solution of Eq. (4) on this path can be shown to be

$$\begin{aligned}
F_{dA1-2} = {} & \frac{1}{2\pi}\sum_{i=1}^{8} \frac{n_1(H^2-R^2) - m_1 H b_i - l_1 a_i (H^2-R^2)}{\sqrt{\left(a_i^2+1\right)(H^2-R^2)^2 + b_i^2 H^2}} \\
& \times \sum_{j=1}^{2} (-1)^j \tan^{-1}\left[H \frac{\left(a_i^2+1\right)\bar{x}_2 + a_i b_i}{\sqrt{\left(a_i^2+1\right)(H^2-R^2)^2 + b_i^2 H^2}} \right] \\
& + \frac{\delta_2}{2\pi}\sum_{i=1}^{2} \frac{x_{Pi} H m_1 - l_1(H^2-R^2)}{\sqrt{x_{Pi}^2 H^2 + (H^2-R^2)^2}} \\
& \times \sum_{j=1}^{2} (-1)^j \tan^{-1}\left[\frac{H\bar{z}_2}{\sqrt{x_{Pi}{}^2 H^2 + (H^2-R^2)^2}} \right] \\
& + \frac{1}{2\pi}\sum_{i=1}^{4}\sum_{j=1}^{2} (-1)^j \Big(n_1\sqrt{H^2-R^2}\,R\cos\theta + m_1 R^2 \theta \\
& - l_1\sqrt{H^2-R^2}\,R\sin\theta \Big)\Big/ H^2
\end{aligned} \tag{6}$$

Here if $n_1 = 1$, $\delta_2 = 1$; otherwise $\delta_2 = 0$. If dA_1 is parallel to the base plane, two of the direction cosines (l_1 and m_1) are zeroes and $n_1 = 1$.

If dA_1 is perpendicular to the base plane and the normal to dA_1 passes through the center of the sphere and a corner of the rectangle, Eq. (6) reduces to the view factor for configuration B-3 minus half the view factor for configuration B-39 (see Ref. 6). Similarly, if dA_1 is parallel to base plane and the sphere is considered at appropriate position, Eq. (6) reduces to the view factor to configuration B4 minus half the view factor to configuration B40 (see Ref. 6).

Conclusions

Closed form solutions are presented for the view factors between a differential area and a finite rectangle with an intervening parallelepiped or sphere, when both the rectangle and the intervening object are kept on a base plane arbitrarily. The expressions presented are applicable irrespective of the dimensions and positions of rectangle and intervening object. However, appropriate limits of integration have to be substituted in the presented closed form solutions, depending on the corresponding constraints. For three specific cases, the limits of integration have been presented along with corresponding constraints. Depending on the positions and dimensions of rectangle and intervening object, a number of subcases are possible, for which the limits of integration and corresponding constraints have not been presented for the sake of brevity. If the parallelepiped or sphere does not intervene the view, the closed form solutions presented reduce to the corresponding particular cases, for which the expressions are available in the literature.

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